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Pancyclic BIBD block-intersection graphs

Aygul Mamut[a,1], David A. Pike[b,*,2], Michael E. Raines[c]

[a] *Department of Mathematics, Xinjiang University, Urumchi, Xinjiang, 830046, People's Republic of China*

[b] *Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Nfld, Canada A1C 5S7*

[c] *Department of Mathematics and Statistics, Western Michigan University, Kalamazoo, MI 49008-5152, USA*

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Dedicated to Curt Lindner on the occasion of his 65th birthday

Abstract

Given a combinatorial design $\mathscr{D}$ with block set $\mathscr{B}$, its block-intersection graph $G_{\mathscr{D}}$ is the graph having vertex set $\mathscr{B}$ such that two vertices b_1 and b_2 are adjacent if and only if b_1 and b_2 have non-empty intersection. In this paper, we prove that if $\mathscr{D}$ is a balanced incomplete block design, BIBD(v,k,λ), with arbitrary index λ, then $G_{\mathscr{D}}$ contains a cycle of each length $\ell = 3,4,\ldots,|V(G_{\mathscr{D}})|$.

Keywords: Block designs; Block-intersection graphs; Cycles of every length

A graph G, with vertex set $V(G)$, is said to be *pancyclic* if it contains a cycle of length ℓ for each $\ell \in \{3,4,\ldots,|V(G)|\}$. In the 1970s Bondy formulated the "metaconjecture" that whenever an hypothesis $\mathscr{H}$ implies that a graph is Hamiltonian (i.e. the graph contains a cycle of length $|V(G)|$), then $\mathscr{H}$ also implies that the graph is pancyclic, except perhaps for a small number of exceptional cases [5]. Several results that support this metaconjecture have since been proved (see, for instance, [6,13–15]).

In 1972, Chvátal and Erdős published the now famous result that if a graph G has vertex connectivity at least as large as α, where we let α denote the cardinality of a maximum set of independent vertices in G, then G is Hamiltonian [7]. Efforts to apply Bondy's metaconjecture to the Chvátal–Erdős condition have yielded few results, limited to cases in which $\alpha \leqslant 3$ [3,12].

In this paper, we consider the class of graphs consisting of block-intersection graphs of balanced incomplete block designs. As our main result we show that these graphs, which satisfy the Chvátal–Erdős condition, are pancyclic.

A *balanced incomplete block design* of order v, having block size k and index λ, or BIBD(v,k,λ), is an ordered pair $(V,\mathscr{B})$, where V is a set of v points and $\mathscr{B}$ is a collection of k-subsets of V known as *blocks*, such that every pair of points of V occurs in exactly λ blocks of $\mathscr{B}$. The *block-intersection graph* of such a design $\mathscr{D}$ is the graph $G_{\mathscr{D}}$ having vertex set $\mathscr{B}$, and in which two vertices are adjacent if and only if their corresponding blocks share at least one point of V.

The investigation of cycle properties of block-intersection graphs appears to have been initiated in 1987 by Ron Graham, who posed the question: does every BIBD$(v,3,1)$ have a Hamiltonian block-intersection graph [2]? In 1988, Horák and Rosa showed that the block-intersection graph of any BIBD$(v,k,1)$ satisfies the Chvátal–Erdős condition and hence is

* Corresponding author.

E-mail addresses: aygulmamut@hotmail.com (A. Mamut), dapike@math.mun.ca (D. A. Pike), michael.raines@wmich.edu (M.E. Raines).

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Hamiltonian [11]. Their proof technique is easily extended to arbitrary $\lambda \geqslant 1$, and can also be refined to show that the block-intersection graph of any BIBD(v,k,λ) is Hamilton-connected (see [7, Theorem 3]).

Concerning pancyclism, Alspach and Hare proved that if $\mathscr{D}$ is any BIBD$(v,k,1)$ with $k \geqslant 3$, then each edge of $G_{\mathscr{D}}$ is contained in a cycle of length ℓ, for each $\ell \in \{3,4,\ldots,|V(G_{\mathscr{D}})|\}$ (i.e. $G_{\mathscr{D}}$ is *edge-pancyclic*) [1]; Hare subsequently proved a similar result for pairwise balanced designs [10]. If $\mathscr{D}$ is a BIBD$(v,2,1)$, then by observing that $G_{\mathscr{D}}$ is in fact $L(K_v)$, the line graph of the complete graph K_v, we also know that $G_{\mathscr{D}}$ is pancyclic [14].

As $G_{\mathscr{D}}$ is known to be pancyclic whenever $\mathscr{D}$ is a balanced incomplete block design having index $\lambda = 1$, we hereafter focus on the situation in which $\lambda \geqslant 2$. The technique used in [1] relies on the neighbourhood of each vertex of $G_{\mathscr{D}}$ having a special structure which can only be assured when the index λ is 1. Consequently we now proceed with a new approach.

If e is an edge of $G_{\mathscr{D}}$, say $e=\{b_1,b_2\}$, then clearly $|b_1 \cap b_2| \geqslant 1$. Hence e is contained in some clique of $G_{\mathscr{D}}$ that consists of all of the blocks that share a common point, say v_0, of V, and so we can label e with a value such as v_0 that is representative of a clique that contains e. If $|b_1 \cap b_2| = 1$, then the only possible clique representative for e is the single point of $b_1 \cap b_2$. However, if $|b_1 \cap b_2| \geqslant 2$, then we have the freedom to choose a clique representative for e from one of $|b_1 \cap b_2|$ different values.

We now ask: what is the maximum length t of any cycle in $G_{\mathscr{D}}$ having the property that a distinct clique representative can be selected for each of the t edges of the cycle? Equivalently, what is the maximum length $2t$ of any cycle in the bipartite incidence graph In$(\mathscr{D})$ (as described in [4, Section 5]) that is associated with the design $\mathscr{D}$? We establish a lower bound of 2α on the value of t with the following lemma.

Lemma 1. *Let $\mathscr{D}=(V,\mathscr{B})$ be a* BIBD(v,k,λ) *with $\lambda \geqslant 2$. Let A be a set of independent vertices in $G_{\mathscr{D}}$. Then $G_{\mathscr{D}}$ has a cycle of length $2|A|$, containing each vertex of A, and having the property that a distinct clique representative can be selected for each of the $2|A|$ edges of the cycle.*

Proof. Let $A=\{a_1,a_2,\ldots,a_i\}$ and let $a_{j,1},a_{j,2},\ldots,a_{j,k}$ be the k points of a_j for each $j \in \{1,2,\ldots,i\}$.

First consider the case in which k is odd, say $k=2c+1$ for some integer c. Let $A_1=\{(a_j,a_{j+1}) \mid 1 \leqslant j \leqslant i\}$, where, for convenience, we let $a_{i+1}=a_1$.

Construct a bipartite graph B_1 having vertex set $A_1 \cup (\mathscr{B}-A)$, and in which a vertex (a_j,a_{j+1}) of A_1 is adjacent to a vertex $b \in \mathscr{B}-A$ if and only if $\{a_{j,1},a_{j,2},\ldots,a_{j,c}\} \cap b \neq \emptyset$ and $\{a_{j+1,c+1},\ a_{j+1,c+2},\ldots,a_{j+1,k}\} \cap b \neq \emptyset$. We now seek a matching in B_1 that saturates A_1, since each edge $\{(a_j,a_{j+1}),b\}$ of such a matching will correspond to a 2-path (a_j,b,a_{j+1}) in $G_{\mathscr{D}}$, where the clique representative for the edge $\{a_j,b\}$ is in the set $\{a_{j,1},a_{j,2},\ldots,a_{j,c}\}$ and the clique representative for the edge $\{b,a_{j+1}\}$ is in the set $\{a_{j+1,c+1},a_{j+1,c+2},\ldots,a_{j+1,k}\}$.

To establish the existence of such a matching in B_1, we need only show that Hall's matching condition is satisfied; i.e. that $|S| \leqslant |N(S)|$ for each non-empty subset S of A_1 [9]. Let $S \subseteq A_1$ and let $\sigma=|S|$.

Observe that for each $j \in \{1,2,\ldots,i\}$, the vertex (a_j,a_{j+1}) of A_1 gives rise to $c(c+1)$ pairs of points of the form (v_1,v_2) where $v_1 \in \{a_{j,1},a_{j,2},\ldots,a_{j,c}\}$ and $v_2 \in \{a_{j+1,c+1},a_{j+1,c+2},\ldots,a_{j+1,k}\}$, and that each pair of points occurs λ times among the blocks of $\mathscr{B}-A$. Any block of $\mathscr{B}-A$ that contains such a pair of points is a neighbour of (a_j,a_{j+1}) in B_1. The blocks of $N(S)$ collectively contain all $\sigma\lambda c(c+1)$ such pairs of points that are formed from the σ vertices of S, and since each block holds $\binom{k}{2}$ pairs of points, it follows that

$$|N(S)| \geqslant \frac{\sigma\lambda c(c+1)}{\binom{k}{2}} = \frac{2\sigma\lambda c(c+1)}{k(k-1)} = \frac{\sigma\lambda(c+1)}{k} = \sigma\left(\frac{\lambda(c+1)}{2c+1}\right) \geqslant \sigma\left(\frac{2c+2}{2c+1}\right) > \sigma = |S|,$$

and so Hall's condition is satisfied.

Now let the edges $\{(a_j,a_{j+1}),b_j\}$, $j=1,2,\ldots,i$, comprise a matching in B_1 that saturates A_1. Then $(a_1,b_1,a_2,b_2,\ldots,a_i,b_i,a_1)$ is a $2|A|$-cycle in $G_{\mathscr{D}}$ with the desired properties.

The case in which k is even is similar. □

Noting that $G_{\mathscr{D}}$ satisfies the Chvátal–Erdős condition, it immediately follows from Dirac's fan lemma [8] that if $x \in V(G_{\mathscr{D}})=\mathscr{B}$ and S is any subset of $\mathscr{B}$ such that $|S| \geqslant \alpha$, then in $G_{\mathscr{D}}$ there exists a set of α paths from x to S such that each pair of paths has only the vertex x in common. However, Dirac's fan lemma does not specify the lengths of these paths. As we shall want paths of length 2 (as well as additional properties) when proving our main result, we now present the following lemma.

Lemma 2. *Let $\mathscr{D}=(V,\mathscr{B})$ be a* BIBD(v,k,λ) *with $\lambda \geqslant 2$. Let A be a set of independent vertices in $G_{\mathscr{D}}$, and let $x \in \mathscr{B}-A$. Then there exists a set of $|A|$ 2-paths in $G_{\mathscr{D}}$ from x to A, such that each pair of 2-paths has only x in common, and such that each pair's interior vertices are adjacent.*

Proof. Let $i = |A|$ and denote by $a_1, a_2, \ldots, a_i$ the vertices of A.

Let $v_0 \in x$. Construct a bipartite graph B_2 having vertex set $A \cup (\mathscr{B} - (A \cup \{x\}))$, in which a vertex $a_j \in A$ is adjacent to $b \in \mathscr{B} - (A \cup \{x\})$ if and only if $a_j \cap b \neq \emptyset$ and $v_0 \in b$. It now suffices to show that in B_2 there exists a matching that saturates A, since each edge $\{a_j, b\}$ of such a matching in B_2 will correspond to a 2-path (a_j, b, x) in $G_{\mathscr{D}}$.

If $v_0 \notin a_j$ for some $a_j \in A$, then among the blocks of $\mathscr{B} - A$, each point $v_1 \in a_j$ must be paired with v_0 exactly λ times. If instead $v_0 \in a_j$, then each point $v_1 \in a_j - \{v_0\}$ must be paired with v_0 exactly $\lambda - 1$ times among the blocks of $\mathscr{B} - A$. Note that since each such pair of points contains the point v_0, then each block of $\mathscr{B} - A$ can contain at most $k - 1$ such pairs.

Let $S \subseteq A$ and let $\sigma = |S|$. Since v_0 can be contained in at most one block of A, the vertices of S therefore give rise to at least $(\sigma - 1)\lambda k + (\lambda - 1)(k - 1)$ pairs of points that each contain v_0 and which must be contained in blocks of $N(S) \cup \{x\}$. It follows that

$$|N(S)| \geqslant \frac{(\sigma - 1)\lambda k + (\lambda - 1)(k - 1)}{k - 1} - 1 = \frac{(\sigma - 1)\lambda k}{k - 1} + \lambda - 2 > (\sigma - 1)\lambda + \lambda - 2 = \sigma\lambda - 2.$$

Hence $|N(S)| \geqslant \sigma\lambda - 1 \geqslant 2\sigma - 1 \geqslant \sigma = |S|$. Since Hall's matching condition is satisfied, B_2 has a matching that saturates A. $\square$

We are now prepared for the main result.

Theorem 1. *Let $\mathscr{D} = (V, \mathscr{B})$ be a* BIBD(v, k, λ) *with $\lambda \geqslant 2$. Then $G_{\mathscr{D}}$ is pancyclic.*

Proof. First observe that the replication number of $\mathscr{D}$ is $r = (\lambda(v - 1))/(k - 1) \geqslant \lambda v/k \geqslant \lambda\alpha$. Since $\lambda \geqslant 2$, each clique in $G_{\mathscr{D}}$ that is induced by each vertex of V has order at least 2α, and so $G_{\mathscr{D}}$ contains a cycle of each length $\ell \in \{3, 4, \ldots, 2\alpha\}$.

Let $A = \{a_1, a_2, \ldots, a_\alpha\}$ be a maximum set of independent vertices in $G_{\mathscr{D}}$, and suppose that C is a cycle in $G_{\mathscr{D}}$ that contains each vertex of A. If C is not a Hamilton cycle, then we can proceed to construct a cycle C' having one edge more than C, such that C' contains each vertex of A. By initially using Lemma 1 to select such a cycle C of length 2α, and then iterating this process, we establish that $G_{\mathscr{D}}$ is pancyclic.

So, given such a cycle C that is not a Hamilton cycle and which contains each vertex of A, let $x \in \mathscr{B}$ be a vertex not contained in the cycle C. Using Lemma 2, we obtain a set $\{(x, b_i, a_i) \mid 1 \leqslant i \leqslant \alpha\}$ of α 2-paths in $G_{\mathscr{D}}$ such that each pair of 2-paths has only the vertex x in common, and such that $b_i \cap b_j \neq \emptyset$ for each $\{i, j\} \subseteq \{1, 2, \ldots, \alpha\}$. Some of the interior vertices of these 2-paths may be contained within C, and so without loss of generality we assume that each vertex of $\{b_1, b_2, \ldots, b_\mu\}$ is contained in C while each vertex of $\{b_{\mu+1}, b_{\mu+2}, \ldots, b_\alpha\}$ is not contained in C.

Fix an orientation of the cycle C and, referring to this orientation, let z^+ denote the vertex of C subsequent to the vertex z of C, and let z^{+2} denote the vertex of C subsequent to the vertex z^+. Now consider the set $S = \{x, b_1^+, b_2^+, \ldots, b_\mu^+, a_{\mu+1}^{+2}, a_{\mu+2}^{+2}, \ldots, a_\alpha^{+2}\}$ of vertices in $G_{\mathscr{D}}$. Clearly $b_i^+ \neq b_j^+$ whenever $i \neq j$, and also $a_i^{+2} \neq a_j^{+2}$ whenever $i \neq j$. And for each $i \in \{1, 2, \ldots, \alpha\}$, $x \neq b_i^+$ and $x \neq a_i^{+2}$.

If $b_i^+ = a_j^{+2}$ for some $i \in \{1, 2, \ldots, \mu\}$ and some $j \in \{\mu + 1, \mu + 2, \ldots, \alpha\}$, then we can form the cycle C' by removing from C the edge $\{a_j, b_i\}$ and replacing it with the 2-path (a_j, b_j, b_i).

Otherwise, we may now assume that $|S| = \alpha + 1$. Hence S cannot be a set of independent vertices and so there must exist an edge between some pair of vertices of S. Several cases result:

(1) If x and b_i^+ are adjacent, where $1 \leqslant i \leqslant \mu$, then we construct C' by removing the edge $\{b_i, b_i^+\}$ from C and replacing it with the 2-path (b_i, x, b_i^+).
(2) If x and a_i^{+2} are adjacent, where $\mu < i \leqslant \alpha$, then we construct C' by removing the 2-path (a_i, a_i^+, a_i^{+2}) from C and replacing it with the 3-path (a_i, b_i, x, a_i^{+2}).
(3) If b_i^+ and b_j^+ are adjacent, where $1 \leqslant i \leqslant \mu$ and $1 \leqslant j \leqslant \mu$, then we construct C' by removing from C the two edges $\{b_i, b_i^+\}$ and $\{b_j, b_j^+\}$, and inserting the edge $\{b_i^+, b_j^+\}$ as well as the 2-path (b_i, x, b_j).
(4) If b_i^+ and a_j^{+2} are adjacent, where $1 \leqslant i \leqslant \mu$ and $\mu < j \leqslant \alpha$, then we remove the edge $\{b_i, b_i^+\}$ and the 2-path (a_j, a_j^+, a_j^{+2}) from C, and insert the edge $\{b_i^+, a_j^{+2}\}$ and the 3-path (b_i, x, b_j, a_j).
(5) If a_i^{+2} and a_j^{+2} are adjacent, where $\mu < i \leqslant \alpha$ and $\mu < j \leqslant \alpha$, then we delete the two 2-paths (a_i, a_i^+, a_i^{+2}) and (a_j, a_j^+, a_j^{+2}), and insert the edge $\{a_i^{+2}, a_j^{+2}\}$ as well as the 4-path (a_i, b_i, x, b_j, a_j).

Note that in each case (even in the cases in which we remove a 2-path from C), each vertex of A is contained within the new cycle C'. $\square$

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